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On magnetohydrodynamic flows generated by an electric current discharge

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Magnetohydrodynamic flows generated in a semi-infinite viscous incompressible conducting fluid by the discharge of an electric current J_0 from a point on the infinite plane bounding the fluid develop singularities when J_0 exceeds a certain critical value. In practical applications sometimes currents much in excess of the critical value of J_0 may be passed between electrodes before singularities appear in the velocity field. In this paper we consider the flow field associated with some current distributions and attempt to provide an explanation for the discrepancy between theory and experiment.

1. Introduction

In some welding problems, such as gas tungsten arc welding, an electric current is passed between two electrodes and the current lines diverge from the smaller electrode, usually the cathode, to the larger one, usually the anode (Eberhart & Seban 1966; Woods & Milner 1971). The Lorentz force set up by this current system and the associated magnetic field is rotational and thus it generates a flow field. Such a flow field has been observed, for example, by Woods & Milner. A very approximate analysis of the mathematics of this problem was performed by Maecker (1955). It must be emphasized, however, that it is the rotational part of the Lorentz force that drives these flows and not pressure gradients, as assumed by some authors. Shercliff (1970) investigated the flow field set up in an inviscid fluid by an electric current supplied through a small hole in the wall bounding the semi-infinite region occupied by the fluid, for the case where the effect of the velocity on the electromagnetic field is assumed negligible.

The velocity field, corresponding to the inviscid case, considered by Shercliff is singular all along the axis of the discharge. It was shown by Sozou (1971) that when the fluid viscosity is taken into consideration the axial singularity in the velocity field disappears. The viscous flow field has a jet-like structure, similar to that occurring in a momentum jet emerging from a small hole in the wall bounding a semi-infinite fluid (Squire 1952), and when $K = 2J_0^2/\pi\rho\nu^2$ exceeds 300·1, it develops singularities and breaks down. Here J_0 is the total current discharged, ρ is the fluid density and ν the coefficient of kinematic viscosity. Thus for a given fluid there is a limit to the magnitude of the current which may be discharged before the velocity field breaks down. The breakdown of the velocity field may be regarded as turbulence. It has been brought to the author's attention that occasionally electric currents of much larger magnitude than $(150\pi\rho\nu^2)^{\frac{1}{2}}$ may be passed between the electrodes before a breakdown of the velocity field is observed. It was shown by Sozou & English (1972) that when the parameter $4\pi\nu\sigma$, where σ is the conductivity of the fluid, is of order unity or larger the value of K at which the velocity field breaks down is considerably increased. In most practical applications, however, the parameter $4\pi\nu\sigma$ is small and we must, therefore, look elsewhere for an adequate explanation of this discrepancy.

This paper is an attempt to explain the discrepancy between theory and experiment. We first consider the case where the current is radial and uniformly distributed within a conical region about the axis of symmetry (and zero outside this region). The rotational part of the Lorentz force generates a velocity field in the conical region occupied by the current. This spreads into the rest of the fluid so that at the interface there is continuity of velocity and stress. Our calculations show that as the angle of the cone in which the current is discharged decreases so does the value of K at which the velocity field becomes singular. This modification, therefore, cannot provide the required explanation. This must be sought in the structure of the current discharge, that is, the force that drives the velocity field.

2. Basic equations of the problem

We consider a uniform incompressible fluid of density ρ and kinematic viscosity ν, occupying the semi-infinite region $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ of a spherical polar co-ordinate system (r, θ, ϕ). At $\theta = \frac{1}{2}\pi$ there is a fixed plane and at the origin there is a current source supplying to the fluid region $0 \leqslant \theta \leqslant \theta_0$ an electric current J_0 per unit time. We consider a steady state and assume that the current is purely radial and symmetric about the line $\theta = 0$, that is, we assume that the current density $\mathbf{j}$ is given by

$$\mathbf{j} = \hat{\mathbf{r}} J_0 f'(\mu)/2\pi r^2, \tag{1}$$

where $\mu = \cos\theta$ and a prime denotes differentiation with respect to μ. Such a configuration can, for example, be approximately set up by the discharge of a current, in a plasma or a liquid metal, from an infinitesimally small cathode to a much larger anode, with the plane of the electrodes perpendicular to the line joining their centres (Maecker 1955). Within the discharge, owing to ohmic heating, the temperature, and therefore the conductivity, is higher than in the rest of the fluid and is variable. We need not, therefore, satisfy the equation $\nabla \times \mathbf{j} = 0$.

For such a current system (Sozou 1971)

$$f(1) - f(\mu_0) = 1, \tag{2}$$

where $\mu_0 = \cos\theta_0$, and the magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \times 2J_0[f(1) - f(\mu)]/r(1-\mu^2)^{\frac{1}{2}}, \tag{3}$$

where $f(\mu) = f(\mu_0)$, for $0 \leqslant \mu \leqslant \mu_0$.

The velocity field $\mathbf{v}$, defined in terms of a stream function ψ, is given by

$$\mathbf{v} = \left(-\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}, -\frac{1}{r(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\psi}{\partial r}, 0\right), \tag{4}$$

where
$$\psi = \begin{cases} \nu r G(\mu, K) & (0 \leqslant \mu \leqslant \mu_0), \quad (5a) \\ \nu r g(\mu, K) & (\mu \geqslant \mu_0). \quad (5b) \end{cases}$$

Here $K = 2J_0^2/\pi\rho\nu^2$. On taking the curl of the momentum equation and making use of (1), (3), (4) and (5), we obtain two fourth-order equations, one for $0 \leqslant \mu \leqslant \mu_0$ and one for $\mu \geqslant \mu_0$, which can be integrated three times to give

$$G^2 - 2(1-\mu^2)\,G' - 4\mu G = K(A\mu^2 + B\mu + C), \quad (6a)$$

$$g^2 - 2(1-\mu^2)\,g' - 4\mu g = K[a\mu^2 + b\mu + c - 2F(\mu)], \quad (6b)$$

where A, B, C, a, b and c are constants of integration and F is the expression obtained by integrating $[f(1) - f(\mu)]f'(\mu)/(1-\mu^2)$ three times. The pressure p, obtained by integrating the momentum equation, apart from an additive constant, is given by

$$p = \nu^2 P(\mu)/r^2,$$

where
$$P = \begin{cases} -\tfrac{1}{2}G^2/(1-\mu^2) - G' - \tfrac{1}{2}KA & (0 \leqslant \mu \leqslant \mu_0), \quad (7a) \\ -\tfrac{1}{2}g^2/(1-\mu^2) - g' - \tfrac{1}{2}K(a - F'') & (\mu \geqslant \mu_0). \quad (7b) \end{cases}$$

The boundary conditions, enabling us to determine a, b, c, A, B and C, are zero velocity on $\mu = 0$, that is $C = 0$, *finite* velocity along the axis $\mu = 1$, that is

$$a + b + c - 2F(1) = 0, \quad (8)$$

$$2a + b - 2F'(1) = 0, \quad (9)$$

and continuity of velocity and stress at the interface, that is continuity of ψ, $\partial\psi/\partial\mu$, $\partial^2\psi/\partial\mu^2$ and P at $\mu = \mu_0$. These conditions imply that

$$a\mu_0^2 + b\mu_0 + c - 2F(\mu_0) = A\mu_0^2 + B\mu_0, \quad (10)$$

$$2a\mu_0 + b - 2F'(\mu_0) = 2A\mu_0 + B, \quad (11)$$

$$a - F''(\mu_0) = A. \quad (12)$$

By the substitution

$$G = -2(1-\mu^2)\,u'/u, \quad g = -2(1-\mu^2)\,u'/u, \quad (13)$$

(6a) and (6b) are transformed into

$$u'' = \frac{K}{4(1-\mu^2)^2}(A\mu^2 + B\mu)\,u, \quad (14a)$$

$$u'' = \frac{K}{4(1-\mu^2)^2}(a\mu^2 + b\mu + c - 2F)\,u. \quad (14b)$$

In (14a) we have set $C = 0$. We solve (14a), by forward integration, subject to the boundary conditions $u(0) = 1$ and $u'(0) = 0$. We assume that u and u' are continuous at $\mu = \mu_0$ and thence proceed to $\mu = 1$ with the solution of (14b).

For a given current distribution, that is, for a given $f(\mu)$, we can determine $F(\mu)$ and from (8)–(12) the constants a, b, c, A and B. A numerical solution of (14) determines the flow field completely. For a current that is uniformly

θ_0	... 90	85	75	60	45	30	15	5	3
K_c	... 300·1	251·0	177·9	117·4	81·3	71·5	63·6	61·4	61·2

TABLE 1. Values of $K_c(\theta_0)$ for some θ_0.

distributed within the cone, that is, for the case where the ohmically heated discharge region has acquired constant temperature and conductivity,

$$f = \mu/(1-\mu_0),\quad F = \frac{1}{2}\left(\frac{1+\mu}{1-\mu_0}\right)^2 \log(1+\mu),\quad a = \tfrac{1}{2}+[\tfrac{3}{2}+\log(1+\mu_0)]/(1-\mu_0)^2,$$

$$b = -1+\left[-1+2\log\left(\frac{4}{1+\mu_0}\right)\right]\Big/(1-\mu_0)^2,\quad c = \tfrac{1}{2}+[-\tfrac{1}{2}+\log(1+\mu_0)]/(1-\mu_0)^2,$$

$$A = \tfrac{1}{2},\quad B = \left[-3+4\mu_0-\mu_0^2+4\log\left(\frac{2}{1+\mu_0}\right)\right]\Big/(1-\mu_0)^2.$$

We have solved (14) for this case, determining for a particular θ_0 the corresponding critical value of K, $K_c(\theta_0)$. For a particular θ_0 when K exceeds $K_c(\theta_0)$ the velocity field develops singularities. These singularities appear initially on the axis of symmetry and as K increases they are pushed into the rest of the fluid region (Sozou 1971). Table 1 shows values of K_c for some θ_0. Inspection of table 1 shows that as θ_0 decreases so does the corresponding critical value of K.

It is easy to show that $K_c(\theta_0) \to 0$ as $\theta_0 \to 0$. When $\mu_0 \to 1$, the constant $B \to -\frac{1}{2}$ and (14a) reduces to

$$u'' = -K\mu u/8(1-\mu)(1+\mu)^2. \tag{15}$$

It can be shown that near $\mu = 1$ one of the solutions of the above equation is a power series of the form $a_1(1-\mu)+a_2(1-\mu)^2+\ldots$, and the other solution behaves like $1-\frac{1}{32}K(1-\mu)\log(1-\mu)$. When we make use of these solutions and (13) we deduce that $G'(\mu) \to \infty$ as μ_0, $\mu \to 1$.

3. Discussion

It is obvious, from table 1, that the above model cannot explain why in practice the velocity field breaks down at $K_c \gg 300$. The intensity, and thus also the breakdown, of the flow field is directly related to the force generating it. The discrepancy concerning the velocity breakdown between the experimental and theoretical model must be ascribed to this force. In practice these discharges consist of an approximately cylindrical column of almost axial current, about the centre-line of the electrodes, surrounded by a more diffuse current having a significant non-axial component. Just below the anode the discharge broadens and assumes a bell shape (Nestor 1962; Eberhart & Seban 1966). The current in the central column and its associated magnetic field produce an almost irrotational component of Lorentz force which is balanced by the fluid pressure. Thus the rotational component of the Lorentz force that drives the velocity field is much less than that corresponding to a case where the current is purely radial such as that considered in the previous section.

The main difference between theory and experiment is the assumption of a point source and a radial current uniform within a cone; this can be illustrated by the following simple example. Consider an electric discharge J_0 between two equal circular electrodes perpendicular to the line joining their centres (in effect a cylindrical column of electrical current). In this case the $\mathbf{J}\times\mathbf{B}$ force is irrotational and it simply modifies the hydrostatic pressure; it does not cause fluid motion. Now imagine that one of the electrodes is slightly larger than the other. The current lines will diverge a little and a small component of the $\mathbf{J}\times\mathbf{B}$ force will be rotational. This will induce a weak velocity field. It might look as though this is at variance with the theory of the previous section, showing that the velocity field generated by a current discharged in a semi-infinite fluid, into a conical region of small semi-vertical angle, will have singularities unless J_0 is relatively small. This discrepancy is, of course, due to the fact that this configuration is substantially different from that of the theory, which is based on a current radiating from a point source into a conical region. If the smaller electrode is reduced in size the current lines will be more divergent and the rotational part of the $\mathbf{J}\times\mathbf{B}$ force and the associated velocity field will increase. This was confirmed experimentally by Woods & Milner. In one experiment they observed the velocity induced by various currents when their two electrodes had diameters of 6 and 15 mm respectively. When they repeated the experiment with the size of the small electrode decreased to 3 mm they found that the observed maximum velocity at any given current was about doubled. The more, of course, the small electrode is decreased the better the configuration set up will fit the theoretical model.

It would be very difficult to construct an exact solution of the nonlinear problem occurring in practice, and the velocity breakdown is caused by the nonlinear (inertia) terms. In order to indicate more clearly the interaction of the central column of current with that diverging between the electrodes and at the same time conserve our similarity method, we consider very briefly the case where J_0 is discharged partly as current radiating from the origin in the fluid region $\mu \geqslant \mu_0$ and partly as a circular cylindrical column of current, which we represent by a line current I along $\theta = 0$. For this case the radial current will be given by

$$\mathbf{j} = \hat{\mathbf{r}}(J_0 - I)f'(\mu)/2\pi r^2, \tag{16}$$

the magnetic field by

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \times 2\{I + (J_0 - I)[f(1) - f(\mu)]\}/r(1-\mu^2)^{\frac{1}{2}} \tag{17}$$

and, in the region $\mu > \mu_0$,

$$\nabla \times (\mathbf{j}\times\mathbf{B}) = \hat{\boldsymbol{\phi}} \times 2(J_0 - I)\{I + (J_0 - I)[f(1) - f(\mu)]\}f'/\pi r^4(1-\mu^2)^{\frac{1}{2}}. \tag{18}$$

For simplicity we choose $f = -(1-\mu)^2/(1-\mu_0)^2$ and assume that

$$J_0 - I \ll I(1-\mu_0)^2.$$

Equation (18) can then be approximated by

$$\nabla \times (\mathbf{j}\times\mathbf{B}) = \hat{\boldsymbol{\phi}} \times 4(J_0 - I)\, I(1-\mu)/\pi(1-\mu_0)^2 r^4(1-\mu^2)^{\frac{1}{2}}. \tag{19}$$

The analysis of the previous section holds for this model and the only difference is that K must now be defined not by $2J_0^2/\pi\rho\nu^2$ but by

$$K = 4(J_0 - I)\, I/\pi\rho\nu^2(1-\mu_0)^2 \ll 2J_0^2/\pi\rho\nu^2.$$

It is easy to see that for this particular model, where most of the current is discharged along the axis, a region of fluid about the axis is occupied by a stronger magnetic field than that corresponding to the model of §2. Yet, as explained above, since the major part of the Lorentz force associated with this current system is irrotational the force driving the velocity field is substantially smaller than that corresponding to the model of §2. Thus, for a given fluid (given ρ and ν) and a given angle of discharge, the critical value of J_0^2 for velocity breakdown will be much larger than that derived from table 1. In the special case $J_0 = I$ the right-hand side of (18) is zero and no flow is generated.

We suggest that the flow generated by the discharged current behaves approximately as described in the preceding section but, since only part of the total current contributes to the force driving the velocity field, the fundamental parameter $J_0^2/\pi\rho\nu^2$ of the problem must be multiplied by a small factor $\lambda\,(<1)$. The value of λ depends on the size and distance between the electrodes and must be determined experimentally. This suggestion is in agreement with observations by Woods & Milner. These authors made experiments with liquid metals and found that the intensity of the flow field is proportional to J_0^2. When the discharged current was of the order of 30 A, they observed a double circulation (in an axial section, one eddy on either side of the centre-line of the electrodes). There was considerable difference in the intensity of the motion with different metals. This is to be expected since for a given J_0 the velocity depends on the fluid density and kinematic viscosity. As the current was increased to the order of 100 A the double circulation gave way to pure rotation which became more rapid at higher currents. This may be regarded as the breakdown envisaged by our theory.

The finite dimensions of the container in which the experiments were carried out, in contrast to the theoretical model, which was formulated for a semi-infinite fluid, influence the overall structure of the velocity field. In practice the fluid particles circulate within the container, forming closed streamlines. In the theoretical model fluid is continuously drawn in from the region adjacent to the plane and discharged from the region about the axis, with the fluid particles accelerating as they come in and slowing down as they pass through the point of their closest approach to the origin. As J_0 increases so does the flow velocity. The eddy field in the container readjusts and intensifies, whereas in the theoretical model the fluid particles are able to penetrate closer to the source before being deflected into the axial region, with the region of inflow increasing and that of outflow decreasing.

When the flow field generated is weak, so that, in the momentum equation, the inertia terms are negligible, a very simple solution to this problem can be constructed. It is easy to show that for the problem considered in the preceding section the linear solution is

$$G = \frac{K}{2(1-\mu_0)^2}\left[A\mu^2 + B\mu + C + D(1-\mu^2)\log\left(\frac{1-\mu}{1+\mu}\right)\right] \quad (\mu_0 \geqslant \mu \geqslant 0),$$

$$g = \frac{K}{2(1-\mu_0)^2}[a\mu^2 + b\mu + c + (1+\mu)\log(1+\mu)] \quad (1 \geqslant \mu \geqslant \mu_0),$$

where the constants A, B, C, D, a, b and c are obtained from the boundary conditions of zero velocity along $\mu = 0$, that is, $G(0) = G'(0) = 0$); continuity of velocity and stress at $\mu = \mu_0$, that is, at $\mu = \mu_0$

$$g = G, \quad g' = G', \quad g'' = G'', \quad g''' = G''';$$

and finite velocity along $\mu = 1$, that is, $g(1) = 0$. After a little algebra these conditions give

$$A = \tfrac{1}{2}(3-4\mu_0+\mu_0^2)+2\log\left[\tfrac{1}{2}(1+\mu_0)\right], \quad B = -\tfrac{1}{4}(1-\mu_0)^2, \quad C = 0, \quad D = \tfrac{1}{2}B,$$

$$a = \tfrac{1}{4}(4-7\mu_0+2\mu_0^2)+2\log\left(\frac{1+\mu_0}{2}\right)+\tfrac{1}{8}(1-\mu_0)^2\log\left(\frac{1-\mu_0}{1+\mu_0}\right),$$

$$b = -\tfrac{1}{2}(2-2\mu_0+\mu_0^2)-\log(1+\mu_0),$$

$$c = \tfrac{3}{4}\mu_0-\log(1+\mu_0)-\tfrac{1}{8}(1-\mu_0)^2\log\left(\frac{1-\mu_0}{1+\mu_0}\right).$$

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